Positive Toeplitz operators on the Bergman space of a minimal bounded homogeneous domain

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Abstract : Necessary and sufficient conditions for positive Toeplitz operators on the Bergman space of a minimal bounded homogeneous domain to be bounded or compact are described in terms of the Berezin transform, the averaging function and the Carleson property.

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1 Introduction

In 1988, Zhu obtained the conditions in order that a positive Toeplitz operator is bounded or compact on the Bergman space of a bounded symmetric domain in its Harish-Chandra realization [11]. In this paper, we extend this result for the case that the domain is a minimal bounded homogeneous domain.

Let D be a bounded homogeneous domain in $\mathbb{C}^n$, $dV(z)$ the Lebesgue measure, $\mathcal{O}(D)$ the space of all holomorphic functions on D, and $L^2_a(D)$ the Bergman space $L^2(D, dV) \cap \mathcal{O}(D)$ of D. We denote by K_D the Bergman kernel of D, that is, the reproducing kernel of $L^2_a(D)$. It is known that $\mathcal{U}$ is a minimal domain with a center t if and only if $K_{\mathcal{U}}(z,t) = K_{\mathcal{U}}(t,t)$ for any $z \in \mathcal{U}$ (see [9, Theorem 3.1]). For example, the open unit disk $\mathbb{D}$, the open unit ball $\mathbb{B}^n$ and the bidisk $\mathbb{D} \times \mathbb{D}$ are minimal domains. It is known that every bounded homogeneous domain is biholomorphic to a minimal bounded homogeneous domain (see [7]).

Let μ be a complex Borel measure on $\mathcal{U}$. The Toeplitz operator T_μ with symbol μ is defined by

$$T_\mu f(z) := \int_{\mathcal{U}} K_{\mathcal{U}}(z,w) f(w)\, d\mu(w) \quad (z \in \mathcal{U}).$$

If $d\mu(w) = u(w)dV(w)$ holds for some $u \in L^\infty(\mathcal{U})$, we have $T_\mu f = P(uf)$, where P is the orthogonal projection from $L^2(\mathcal{U})$ onto $L^2_a(\mathcal{U})$. Therefore, T_μ is a bounded operator on $L^2_a(\mathcal{U})$ with $\|T_\mu\| \leq \|u\|_\infty$. We consider the condition of μ that T_μ is a bounded (or compact) operator on $L^2_a(\mathcal{U})$.

A Toeplitz operator is called positive if its symbol is positive. A result on positive Toeplitz operator of a bounded symmetric domain was obtained in [11].

Zhu proved that the boundedness of the positive Toeplitz operator on $L^2_a(\Omega)$ is equivalent to the boundedness of the Berezin transform $\widetilde{\mu}$ or the averaging function $\widehat{\mu}$ on Ω. The key lemma is [3, Lemma 8]. The proof of this lemma is based on some characteristic properties of a bounded symmetric domain in its Harish-Chandra realization. It is difficult to generalize directly their argument for a bounded homogeneous domain, which is not necessarily symmetric. However, the following theorem enables us to prove the same key estimate (Lemma 3.3) for the Bergman kernel of a minimal bounded homogeneous domain.

Theorem 1.1 ([7, Theorem 1.1])**.** *Let $\mathcal{U} \subset \mathbb{C}^n$ be a minimal bounded homogeneous domain. Take any $\rho > 0$. Then, there exists $C_\rho > 0$ such that*

$$C_\rho^{-1} \leq \left| \frac{K_{\mathcal{U}}(z,a)}{K_{\mathcal{U}}(a,a)} \right| \leq C_\rho$$

for all $z, a \in \mathcal{U}$ with $\beta(z,a) \leq \rho$, where β denotes the Bergman distance on $\mathcal{U}$.

Using Lemma 3.3 and Zhu's method (see [11] or [12]), we deduce a certain relation of averaging functions to the Carleson measures (Theorem 3.7). Moreover, we obtain the following theorem.

Theorem 1.2. *Let $\mathcal{U} \subset \mathbb{C}^n$ be a minimal bounded homogeneous domain and μ a positive Borel measure on $\mathcal{U}$. Then the following conditions are all equivalent.*
(a) T_μ is a bounded operator on $L^2_a(\mathcal{U})$.
(b) The Berezin transform $\widetilde{\mu}(z)$ is a bounded function on $\mathcal{U}$.
(c) For all $p \geq 1$, μ is a Carleson measure for $L^p_a(\mathcal{U})$.
(d) The averaging function $\widehat{\mu}(z)$ is bounded on $\mathcal{U}$.

The representative domain of the tube domain over the Vinberg's cone is an example of nonsymmetric minimal bounded homogeneous domain. Theorem 1.2 generalizes Zhu's result ([11, Theorem A]) to such domain, for instance.

In the part $(c) \implies (a)$, we use the boundedness of the positive Bergman operator $P^+_{\mathcal{U}}$ on $L^2(\mathcal{U}, dV)$. Using Schur's theorem (see [12, Theorem 3.6]), it is sufficient to find a positive function h and a positive constant C such that

$$\int_{\mathcal{U}} |K_{\mathcal{U}}(z,w)|\, h(w)\, dV(w) \leq Ch(z)$$

holds for all $z \in \mathcal{U}$. If $\mathcal{U}$ is a bounded symmetric domain in its Harish-Chandra realization, we can construct such h and C from the Forelli-Rudin inequalities (see [12, Theorem 7.5], [4, Proposition 8]). But it is difficult to do this on minimal bounded homogeneous domains. Instead, we make use of the boundedness of the positive Bergman operator $P^+_{\mathcal{D}}$ on $L^2(\mathcal{D}, dV)$, where $\mathcal{D}$ is a homogeneous Siegel domain of type II ([2, Theorem II.7]). Since every bounded homogeneous domain is biholomorphic to some Siegel domain, we deduce the boundedness of $P^+_{\mathcal{U}}$ (see section 2.4).

To prove the compactness of T_μ, we consider a vanishing Carleson measure for $L^2_a(\mathcal{U})$. We know that $K_{\mathcal{U}}(a,a) \to \infty$ as $a \to \partial\mathcal{U}$ (see [8, Proposition 5.2]). Therefore, we can prove Theorem 3.10 in the same way as in [12, Theorem 7.7]. We obtain the condition of the compactness of the Toeplitz operator.

Theorem 1.3. *Let $\mathcal{U} \subset \mathbb{C}^n$ be a minimal bounded homogeneous domain and μ a finite positive Borel measure on $\mathcal{U}$. Then the following conditions are all equivalent.*
(a) T_μ is a compact operator on $L^2_a(\mathcal{U})$.
(b) The Berezin transform $\widetilde{\mu}(z)$ tends to 0 as $z \to \partial\mathcal{U}$.
(c) μ is a vanishing Carleson measure for $L^2_a(\mathcal{U})$.
(d) The averaging function $\widehat{\mu}(z)$ tends to 0 as $z \to \partial\mathcal{U}$.

2 Preliminaries

2.1 Minimal domain

Let D be a bounded domain in $\mathbb{C}^n$. We say that D is a minimal domain with a center $t \in D$ if the following condition is satisfied: for every biholomorphism $\psi : D \longrightarrow D'$ with $\det J(\psi, t) = 1$, we have

$$\mathrm{Vol}\,(D') \geq \mathrm{Vol}\,(D).$$

From [6, Proposition 3.6] or [9, Theorem 3.1], we see that D is a minimal domain with a center t if and only if

$$K_D(z,t) = \frac{1}{\mathrm{Vol}\,(D)}$$

for any $z \in D$.

The representative bounded homogeneous domain is a generalization of the Harish-Chandra realization for a bounded symmetric domain. Indeed, every bounded homogeneous domain is biholomorphic to a representative bounded homogeneous domain. It is known that any representative bounded homogeneous domain is a minimal domain with a center 0 (see [6, Proposition 3.8]). Therefore, every bounded homogeneous domain is biholomorphic to a minimal bounded homogeneous domain.

2.2 Berezin symbol

We fix a minimal bounded homogeneous domain $\mathcal{U}$ with a center t. For a bounded linear operator T on $L^2_a(\mathcal{U})$, the Berezin symbol $\widetilde{T}$ of T is defined by

$$\widetilde{T}(z) := \langle T k_z, k_z \rangle \quad (z \in \mathcal{U}).$$

For a Borel measure μ on $\mathcal{U}$, we define a function $\widetilde{\mu}$ on $\mathcal{U}$ by

$$\widetilde{\mu}(z) := \int_{\mathcal{U}} |k_z(w)|^2 \, d\mu(w),$$

which is called the Berezin symbol of the measure μ. Since $|K_{\mathcal{U}}(z,w)|$ is a bounded function on $B(t,\rho) \times \mathcal{U}$ (see [7, Proposition 6.1]), $\widetilde{\mu}$ is a continuous function if μ is finite.

Suppose that the Toeplitz operator T_μ is a bounded operator on $L^2_a(\mathcal{U})$. We have

$$\widetilde{T_\mu}(z) = \langle T_\mu k_z, k_z\rangle = \frac{1}{K_{\mathcal{U}}(z,z)^{1/2}} T_\mu k_z(z)$$

by the definition of the reproducing kernel. The right hand side equals

$$\frac{1}{K(z,z)^{1/2}} \int_{\mathcal{U}} K_{\mathcal{U}}(z,w) k_z(w)\, d\mu(w) = \int_{\mathcal{U}} |k_z(w)|^2\, d\mu(w).$$

Therefore, we have

$$\widetilde{T_\mu}(z) = \widetilde{\mu}(z). \tag{2.1}$$

2.3 Carleson measure and vanishing Carleson measure

Let μ be a positive Borel measure on $\mathcal{U}$ and $p \geq 1$. We say that μ is a Carleson measure for $L^p_a(\mathcal{U})$ if there exists a constant $M > 0$ such that

$$\int_{\mathcal{U}} |f(z)|^p\, d\mu(z) \leq M \int_{\mathcal{U}} |f(z)|^p\, dV(z)$$

for all $f \in L^p_a(\mathcal{U})$. It is easy to see that μ is a Carleson measure for $L^p_a(\mathcal{U})$ if and only if $L^p_a(\mathcal{U}) \subset L^p_a(\mathcal{U}, d\mu)$ and the inclusion map

$$i_p : L^p_a(\mathcal{U}) \longrightarrow L^p_a(\mathcal{U}, d\mu)$$

is bounded.

Suppose μ is a Carleson measure for $L^2_a(\mathcal{U})$. We say that μ is a vanishing Carleson measure for $L^2_a(\mathcal{U})$ if the inclusion map

$$i_2 : L^2_a(\mathcal{U}) \longrightarrow L^2_a(\mathcal{U}, d\mu)$$

is compact.

2.4 Boundedness of the positive Bergman operator

In order to prove the part $(c) \Longrightarrow (a)$ in Theorem 1.2, we use the boundedness of the positive Bergman operator $P^+_{\mathcal{U}}$ on $L^2(\mathcal{U}, dV)$ defined by

$$P^+_{\mathcal{U}} g(z) := \int_{\mathcal{U}} |K_{\mathcal{U}}(z,w)|\, g(w)\, dV(w) \tag{2.2}$$

for $g \in L^2(\mathcal{U}, dV)$. We prove that $P^+_{\mathcal{U}}$ is a bounded operator on $L^2(\mathcal{U}, dV)$.

It is known that every bounded homogeneous domain is holomorphically equivalent to a homogeneous Siegel domain [10]. Let Φ be a biholomorphic map from $\mathcal{U}$ to a Siegel domain $\mathcal{D}$. We define a unitary map U_Φ from $L^2(\mathcal{U}, dV)$ to $L^2(\mathcal{D}, dV)$ by

$$U_\Phi f(\zeta) := f(\Phi^{-1}(\zeta)) \left|\det J(\Phi^{-1}, \zeta)\right|.$$

Then, we have

$$U_\Phi \circ P^+_{\mathcal{U}} = P^+_{\mathcal{D}} \circ U_\Phi \quad (f \in L^2(\mathcal{U}, dV)).$$

Therefore, the boundedness of $P^+_{\mathcal{U}}$ on $L^2(\mathcal{U}, dV)$ is equivalent to the boundedness of $P^+_{\mathcal{D}}$ on $L^2(\mathcal{D}, dV)$. On the other hand, Békollé-Kagou proved the boundedness of the positive Bergman operator $P^+_{\mathcal{D}}$ on $L^2(\mathcal{D}, dV)$ ([2, Theorem II.7]). Therefore, we have the following lemma.

Lemma 2.1. *The operator $P^+_{\mathcal{U}}$ is bounded on $L^2(\mathcal{U}, dV)$.*

3 Some Lemmas

In this section, we show some lemmas for a minimal bounded homogeneous domain $\mathcal{U}$ with a center $t \in \mathcal{U}$. Although the proofs of these lemmas are almost same as the ones for the case of symmetric domain ([3],[1],[12]), we write them here for the sake of completeness. In this section, $K(z, w)$ means $K_{\mathcal{U}}(z, w)$. First, we present the following theorem, which plays fundamental roles in this work.

Theorem 3.1 ([7, Theorem A])**.** *For any $\rho > 0$, there exists $C_\rho > 0$ such that*

$$C_\rho^{-1} \le \left| \frac{K(z, a)}{K(a, a)} \right| \le C_\rho$$

for all $z, a \in \mathcal{U}$ such that $\beta(z, a) \le \rho$.

For $a \in \mathcal{U}$, let φ_a be an automorphism of $\mathcal{U}$ such that $\varphi_a(a) = t$. Using Theorem 3.1, we prove Theorem 3.7. First, we prove some lemmas.

Lemma 3.2. *One has*

$$|\det J(\varphi_a, z)|^2 = \frac{|K(z, a)|^2}{K(t, t)K(a, a)}, \tag{3.1}$$

$$\left|\det J(\varphi_a^{-1}, z)\right|^2 = \frac{K(t, t)K(a, a)}{|K(\varphi_a^{-1}(z), a)|^2}, \tag{3.2}$$

where $\det J(\varphi_a, z)$ *is the complex Jacobian of φ_a at z.*

Proof. By the transformation formula of the Bergman kernel, we have

$$K(z, a) = K(\varphi_a(z), \varphi_a(a)) \det J(\varphi_a, z)\overline{\det J(\varphi_a, a)}.$$

Since $K(\varphi_a(z), \varphi_a(a)) = K(\varphi_a(z), t) = K(t, t)$, we obtain

$$|\det J(\varphi_a, z)|^2 = \frac{|K(z, a)|^2}{K(t, t)^2 \, |\det J(\varphi_a, a)|^2}. \tag{3.3}$$

On the other hand, we have

$$K(a, a) = K(\varphi_a(a), \varphi_a(a)) \, |\det J(\varphi_a, a)|^2 .$$

This means

$$|\det J(\varphi_a, a)|^2 = \frac{K(a,a)}{K(t,t)}. \tag{3.4}$$

From (3.3) and (3.4), we obtain (3.1). The equality (3.2) follows from

$$\det J(\varphi_a, \varphi_a^{-1}(z)) \det J(\varphi_a^{-1}, z) = 1. \qquad \square$$

For any $z \in \mathcal{U}$ and $\rho > 0$, let

$$B(z,\rho) := \{w \in \mathcal{U} \mid \beta(z,w) \le \rho\}$$

be the Bergman metric disk with center z and radius ρ.

Lemma 3.3 (cf. [3, Lemma 8]). *There exists a constant M_ρ such that*

$$M_\rho^{-1} \le |k_a(z)|^2 \operatorname{Vol}(B(a,\rho)) \le M_\rho$$

for all $a \in \mathcal{U}$ and $z \in B(a,\rho)$.

Proof. Thanks to the invariance of the Bergman distance under biholomorphic transformations, we have

$$\operatorname{Vol}(B(a,\rho)) = \int_{B(t,\rho)} \left|\det J(\varphi_a^{-1}, u)\right|^2 dV(u).$$

By Lemma 3.2, we obtain

$$\begin{aligned} |k_a(z)|^2 \operatorname{Vol}(B(a,\rho)) &= \frac{|K(z,a)|^2}{K(a,a)} \int_{B(t,\rho)} \frac{K(t,t)K(a,a)}{|K(\varphi_a^{-1}(u),a)|^2}\, dV(u) \\ &= K(t,t) \int_{B(t,\rho)} \frac{|K(z,a)|^2}{|K(\varphi_a^{-1}(u),a)|^2}\, dV(u). \end{aligned} \tag{3.5}$$

Since $u \in B(t,\rho)$ means $\beta(t,u) \le \rho$, we have $\beta(a, \varphi_a^{-1}(u)) \le \rho$, so that Theorem 3.1 implies

$$C_\rho^{-1} \le \left| \frac{K(a,a)}{K(\varphi_a^{-1}(u),a)} \right| \le C_\rho. \tag{3.6}$$

On the other hand, we have

$$C_\rho^{-1} \le \left| \frac{K(z,a)}{K(a,a)} \right| \le C_\rho. \tag{3.7}$$

Multiplying (3.6) by (3.7), we obtain

$$C_\rho^{-2} \le \frac{|K(z,a)|}{|K(\varphi_a^{-1}(u),a)|} \le C_\rho^2. \tag{3.8}$$

By (3.5) and (3.8), we complete the proof with $M_\rho = C_\rho^2 K(t,t) \operatorname{Vol}(B(t,\rho))$. $\square$

Since one uses not the symmetry but the homogeneity of a complex domain in the proof of [1, Lemma 5], the following lemma holds for the minimal bounded homogeneous domain $\mathcal{U}$.

Lemma 3.4 ([1, Lemma 5]). *There exists a sequence $\{w_j\} \subset \mathcal{U}$ satisfying the following conditions.*
$(S1)$ $\mathcal{U} = \cup_{j=1}^{\infty} B(w_j, \rho)$.
$(S2)$ $B(w_i, \rho/4) \cap B(w_j, \rho/4) = \emptyset$.
$(S3)$ *There exists a positive integer N such that each point $z \in \mathcal{U}$ belongs to at most N of the sets $B(w_j, 2\rho)$.*

Lemma 3.5 (cf. [1, Lemma 7]). *There exists a constant C such that*

$$|f(a)|^p \leq \frac{C}{\mathrm{Vol}\,(B(a,\rho))} \int_{B(a,\rho)} |f(z)|^p \, dV(z) \tag{3.9}$$

for all $f \in \mathcal{O}(\mathcal{U})$, $p \geq 1$ and $a \in \mathcal{U}$.

Proof. First, we consider the case $a = t$. Since the Bergman metric induces the usual Euclidean topology on $\mathcal{U}$, there exists a Euclidean ball $E(t, R)$ with center t and the radius R such that $E(t, R) \subset B(t, \rho)$. Let f be a holomorphic function on $\mathcal{U}$. Since f has a mean value property, we have

$$f(t) = \frac{1}{\mathrm{Vol}\,(E(t,R))} \int_{E(t,R)} f(z) \, dV(z).$$

Therefore, we have

$$\begin{aligned} |f(t)|^p &\leq \left(\frac{1}{\mathrm{Vol}\,(E(t,R))} \int_{E(t,R)} |f(z)| \, dV(z) \right)^p \\ &\leq \left(\frac{1}{\mathrm{Vol}\,(E(t,R))} \right)^p \left(\|f\|_{L^p(E(t,R))} \, \|1\|_{L^q(E(t,R))} \right)^p, \end{aligned} \tag{3.10}$$

where q denotes the conjugate exponent of p. Since

$$\|1\|^p_{L^q(E(t,R))} = \mathrm{Vol}\,(E(t,R))^{\frac{p}{q}},$$

the last term of (3.10) is equal to

$$(\mathrm{Vol}\,(E(t,R)))^{-p+\frac{p}{q}} \int_{E(t,R)} |f(z)|^p \, dV(z).$$

Therefore, we have

$$|f(t)|^p \leq \frac{1}{\mathrm{Vol}\,(E(t,R))} \int_{E(t,R)} |f(z)|^p \, dV(z)$$

because $-p + \frac{p}{q} = p(-1 + \frac{1}{q}) = 1$.

Now, put $C_R := \frac{1}{\mathrm{Vol}(E(t,R))}$. Note that the constant C_R is independent of p and f. Since $E(t,R) \subset B(t,\rho)$, we have

$$|f(t)|^p \leq C_R \int_{B(t,\rho)} |f(z)|^p \, dV(z). \tag{3.11}$$

Next, we prove the general case. Since $f \circ \varphi_a^{-1}$ is a holomorphic function on $\mathcal{U}$, we have

$$\left| f \circ \varphi_a^{-1}(t) \right|^p \leq C_R \int_{B(t,\rho)} \left| f \circ \varphi_a^{-1}(z) \right|^p \, dV(z) \tag{3.12}$$

by (3.11). Put $w := \varphi_a^{-1}(z)$. Then the inequality (3.12) means

$$|f(a)|^p \leq C_R \int_{B(a,\rho)} |f(w)|^p \, |\det J(\varphi_a, w)|^2 \, dV(w).$$

By Lemma 3.2, the right hand side is equal to

$$C_R \int_{B(a,\rho)} |f(w)|^p \, \frac{|K(w,a)|^2}{K(t,t)K(a,a)} \, dV(w).$$

Therefore we have

$$|f(a)|^p \leq C_R \frac{K(a,a)}{K(t,t)} \int_{B(a,\rho)} |f(w)|^p \left| \frac{K(w,a)}{K(a,a)} \right|^2 \, dV(w). \tag{3.13}$$

By Theorem 3.1, we have

$$C_\rho^{-2} \leq \left| \frac{K(w,a)}{K(a,a)} \right|^2 \leq C_\rho^2 \tag{3.14}$$

on $w \in B(a,\rho)$. Therefore we have

$$|f(a)|^p \leq C_R C_\rho^2 \frac{K(a,a)}{K(t,t)} \int_{B(a,\rho)} |f(w)|^p \, dV(w) \tag{3.15}$$

by (3.13) and (3.14). We see from (3.14) and Lemma 3.3 that

$$C_\rho^{-2} \leq \left| \frac{K(w,a)}{K(a,a)} \right|^2 = \frac{|k_a(w)|^2}{K(a,a)} \leq \frac{M_\rho}{\mathrm{Vol}\,(B(a,\rho)) \; K(a,a)}.$$

Hence we obtain

$$K(a,a) \leq \frac{M_\rho C_\rho^2}{\mathrm{Vol}\,(B(a,r))}. \tag{3.16}$$

By (3.15) and (3.16), we have

$$|f(a)|^p \leq \frac{C}{\mathrm{Vol}\,(B(a,\rho))} \int_{B(a,\rho)} |f(w)|^p \, dV(w).$$

with $C = C_\rho^4 C_R M_\rho K(t,t)^{-1}$. □

Lemma 3.6. *There exists a constant C such that*

$$\sup_{w \in B(a,\rho)} |f(w)|^p \leq \frac{C}{\mathrm{Vol}\,(B(a,\rho))} \int_{B(a,2\rho)} |f(z)|^p \, dV(z)$$

for all $f \in \mathcal{O}(\mathcal{U}), p \geq 1$ and $a \in \mathcal{U}$.

Proof. By Lemma 3.5, there exists a constant C such that

$$|f(w)|^p \leq \frac{C}{\mathrm{Vol}\,(B(w,\rho))} \int_{B(w,\rho)} |f(z)|^p \, dV(z)$$

for any $f \in \mathcal{O}(\mathcal{U})$, $p \geq 1$ and $w \in \mathcal{U}$. Therefore we have

$$\begin{aligned} \sup_{w \in B(a,\rho)} |f(w)|^p &\leq C \sup_{w \in B(a,\rho)} \left(\frac{1}{\mathrm{Vol}\,(B(w,\rho))} \int_{B(w,\rho)} |f(z)|^p \, dV(z) \right) \\ &\leq C \left(\int_{B(a,2\rho)} |f(z)|^p \, dV(z) \right) \sup_{w \in B(a,\rho)} \frac{1}{\mathrm{Vol}\,(B(w,\rho))}, \end{aligned}$$

where the last inequality holds because $B(w,\rho)$ is a subset of $B(a,2\rho)$ for all $w \in B(a,\rho)$. Hence, it is sufficient to prove

$$\sup_{w \in B(a,\rho)} \frac{1}{\mathrm{Vol}\,(B(w,\rho))} \leq \frac{C}{\mathrm{Vol}\,(B(a,\rho))}.$$

Take any $w \in B(a,\rho)$ and let $b \in B(a,\rho) \cap B(w,\rho)$. Then we have

$$\begin{aligned} \mathrm{Vol}\,(B(a,\rho)) &\leq M_\rho \, |k_a(b)|^{-2}, \\ \mathrm{Vol}\,(B(w,\rho)) &\geq M_\rho^{-1} \, |k_w(b)|^{-2} \end{aligned}$$

by Lemma 3.3. Therefore, we obtain

$$\frac{\mathrm{Vol}\,(B(a,\rho))}{\mathrm{Vol}\,(B(w,\rho))} \leq M_\rho^2 \left| \frac{k_w(b)}{k_a(b)} \right|^2. \tag{3.17}$$

On the other hand, we have

$$\begin{aligned} \left| \frac{k_w(b)}{k_a(b)} \right|^2 &= \frac{|K(w,b)|^2}{K(w,w)} \frac{K(a,a)}{|K(a,b)|^2} \\ &= \left| \frac{K(w,a)}{K(w,w)} \right| \left| \frac{K(a,a)}{K(w,a)} \right| \left| \frac{K(w,b)}{K(b,b)} \right|^2 \left| \frac{K(b,b)}{K(a,b)} \right|^2. \end{aligned}$$

Since $\beta(w,a)$, $\beta(w,b)$ and $\beta(a,b)$ do not exceed ρ, we have

$$\left| \frac{k_w(b)}{k_a(b)} \right|^2 \leq C_\rho^6 \tag{3.18}$$

by Theorem 3.1. Therefore, we have

$$\sup_{w \in B(a,\rho)} \frac{1}{\mathrm{Vol}\,(B(w,\rho))} \leq \frac{C}{\mathrm{Vol}\,(B(a,\rho))} \tag{3.19}$$

by (3.17) and (3.18). ∎

By Lemmas 3.3, 3.4 and 3.6, we can prove the following theorem as in the same way of the proof of [11, Theorem 7]. It follows from this theorem that the property of being a Carleson measure is independent of p.

Theorem 3.7 ([11, Theorem 7])**.** *Suppose μ is a positive Borel measure on $\mathcal{U}$ and $p \geq 1$. Then μ is a Carleson measure for $L_a^p(\mathcal{U})$ if and only if*

$$\sup_{a\in\mathcal{U}} \frac{\mu(B(a,\rho))}{\mathrm{Vol}\,(B(a,\rho))} < \infty. \tag{3.20}$$

It is known that $\mathcal{H} := \mathrm{span}\langle K_{\mathcal{U}}(\cdot, w)\rangle_{w\in\mathcal{U}}$ is dense in $L_a^2(\mathcal{U})$. On the other hand, $K_{\mathcal{U}}(\cdot, w)$ is bounded for each $w \in \mathcal{U}$ (see [7, Proposition 6.1]). Therefore $\mathcal{H} \subset H^\infty$, so that H^∞ is dense in $L_a^2(\mathcal{U})$. Since $K(a, a) \to \infty$ as $a \to \partial\mathcal{U}$ (see [8, Proposition 5.2]), we can prove the following lemmas in the same way as in [4].

Lemma 3.8 ([4, Lemma 1])**.** *A sequence $\{k_a\}$ converges to 0 weakly in $L_a^2(\mathcal{U})$ as $a \to \partial\mathcal{U}$.*

Lemma 3.9 ([4, Lemma 5])**.** *Let $\{f_n\}$ be a sequence of functions in $L_a^2(\mathcal{U})$ which is weakly convergent to f. Then $f_n \to f$ uniformly on compact subsets of $\mathcal{U}$.*

From Lemma 3.8 and 3.9, we can prove the following theorem.

Theorem 3.10 ([11, Theorem 11], [12, Theorem 7.7])**.** *Let μ be a finite positive Borel measure on $\mathcal{U}$. Then μ is a vanishing Carleson measure for $L_a^2(\mathcal{U})$ if and only if*

$$\lim_{a\to\partial\mathcal{U}} \frac{\mu(B(a,\rho))}{\mathrm{Vol}\,(B(a,\rho))} = 0.$$

4 Boundedness of the Toeplitz operator

In this section, we prove the main theorem.

Theorem 4.1. *Let $\mathcal{U} \subset \mathbb{C}^n$ be a minimal bounded homogeneous domain and μ a positive Borel measure on $\mathcal{U}$. Then the following conditions are all equivalent.*
(a) T_μ is a bounded operator on $L_a^2(\mathcal{U})$.
(b) $\widetilde{\mu}(z)$ is a bounded function on $\mathcal{U}$.
(c) For all $p \geq 1$, μ is a Carleson measure for $L_a^p(\mathcal{U})$.
(d) $\widehat{\mu}(z)$ is a bounded function on $\mathcal{U}$.

Proof. We have already proved $(c) \iff (d)$ in Theorem 3.7. We will prove $(a) \Longrightarrow (b) \Longrightarrow (d)$ and $(c) \Longrightarrow (a)$.

First, we prove $(a) \Longrightarrow (b)$. Since T_μ is a bounded operator, we have

$$\widetilde{\mu}(z) = \widetilde{T_\mu}(z) = |\langle T_\mu k_z, k_z\rangle| \leq \|T_\mu\|\,\|k_z\|^2 = \|T_\mu\| < \infty,$$

where the first equality follows from (2.1).

Next, we prove $(b) \Longrightarrow (d)$. By Lemma 3.3, we have

$$M_\rho^{-1} \leq |k_z(w)|^2 \,\mathrm{Vol}\,(B(z,\rho)).$$

We integrate this inequality on $B(z,\rho)$ by μ. Then we have

$$M_\rho^{-1} \int_{B(z,\rho)} d\mu(w) \leq \mathrm{Vol}\,(B(z,\rho)) \int_{B(z,\rho)} |k_z(w)|^2 \, d\mu(w).$$

Therefore, we have

$$\begin{aligned} \frac{\mu(B(z,\rho))}{\mathrm{Vol}\,(B(z,\rho))} &\leq M_\rho \int_{B(z,\rho)} |k_z(w)|^2 \, d\mu(w) \\ &\leq M_\rho \, \|k_z\|^2_{L^2(d\mu)} = M_\rho \, \widetilde{\mu}(z). \end{aligned}$$

Therefore we have $\widehat{\mu}(z) \leq M_\rho \, \widetilde{\mu}(z)$, so $\widehat{\mu}(z)$ is a bounded function on $\mathcal{U}$.

Finally, we prove $(c) \Longrightarrow (a)$. For $f \in L^2_a(\mathcal{U})$, we have

$$\begin{aligned} \|T_\mu f\|_2^2 &= \int_{\mathcal{U}} \left| \int_{\mathcal{U}} K_{\mathcal{U}}(z,w) f(w) \, d\mu(w) \right|^2 dV(z) \\ &\leq \int_{\mathcal{U}} \left(\int_{\mathcal{U}} |K_{\mathcal{U}}(z,w)| \, |f(w)| \, d\mu(w) \right)^2 dV(z) \\ &= \int_{\mathcal{U}} \left(\int_{\mathcal{U}} |F_z(w)| \, d\mu(w) \right)^2 dV(z), \end{aligned} \tag{4.1}$$

where we put $F_z(w) := \overline{K_{\mathcal{U}}(z,w)} f(w)$. Since $\overline{K_{\mathcal{U}}(z,\cdot)} \in L^2_a(\mathcal{U})$, we have $F_z \in L^1_a(\mathcal{U})$. Moreover, μ is a Carleson measure. Hence, there exists a positive constant M_μ such that

$$\int_{\mathcal{U}} |F_z(w)| \, d\mu(w) \leq M_\mu \int_{\mathcal{U}} |F_z(w)| \, dV(w). \tag{4.2}$$

By the definition of the Carleson measure, M_μ is independent of z. Therefore, we have

$$\|T_\mu f\|_2^2 \leq M_\mu^2 \int_{\mathcal{U}} \left(\int_{\mathcal{U}} |K_{\mathcal{U}}(z,w)| \, |f(w)| \, dV(w) \right)^2 dV(z) \tag{4.3}$$

by (4.1) and (4.2). Moreover, the right hand side is rewritten as $M_\mu^2 \left\| P^+_{\mathcal{U}} f^+ \right\|_2^2$, where $f^+ = |f|$. Since $P^+_{\mathcal{U}}$ is a bounded operator by Theorem 2.1, we have

$$\|T_\mu f\|_2 \leq M_\mu \left\| P^+_{\mathcal{U}} f^+ \right\|_2 \leq M_\mu \left\| P^+_{\mathcal{U}} \right\| \|f\|_2 .$$

Next, we prove $T_\mu f \in \mathcal{O}(\mathcal{U})$. Since $T_\mu f \in L^2(\mathcal{U})$, it is enough to prove $\langle T_\mu f, g \rangle = 0$ for any $g \in L^2_a(\mathcal{U})^\perp$. We see that

$$\begin{aligned} \langle T_\mu f, g \rangle &= \int_{\mathcal{U}} \left\{ \int_{\mathcal{U}} K_{\mathcal{U}}(z,w) f(w) \, d\mu(w) \right\} \overline{g(z)} \, dV(z) \\ &= \int_{\mathcal{U}} \overline{\left\{ \int_{\mathcal{U}} K_{\mathcal{U}}(w,z) g(z) \, dV(z) \right\}} f(w) \, d\mu(w) \\ &= 0. \end{aligned} \tag{4.4}$$

Note that since

$$\int_{\mathcal{U}}\int_{\mathcal{U}} |K_{\mathcal{U}}(w,z)g(z)f(w)|\, d\mu(w)dV(z) \le M_\mu \left\|P_{\mathcal{U}}^{+}\right\| \|f\|_2 \|g\|_2 < \infty, \tag{4.5}$$

the second equality of (4.4) follows from Fubini's theorem.

Therefore, T_μ is a bounded operator on $L_a^2(\mathcal{U})$. □

5 Compactness of the Toeplitz operator

Suppose $1 < p < \infty$ and q is the conjugate exponent of p. It is known that $(L_a^p(\mathbb{D}))^* \cong L_a^q(\mathbb{D})$ with equivalent norms and under the integral pairing:

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}\, dV(z), \tag{5.1}$$

where $f \in L_a^p(\mathbb{D})$ and $g \in L_a^q(\mathbb{D})$ (see [12, Theorem 4.25]). To prove this, we use the boundedness of the positive Bergman projection $P_{\mathbb{D}}^{+}$ on $L^p(\mathbb{D}, dV)$. But, we do not know that $P_{\mathcal{U}}^{+}$ is a bounded operator on $L^p(\mathcal{U}, dV)$ for $p \neq 2$, whereas the similar statement is shown for homogeneous Siegel domain by Békollé-Kagou. Therefore, we consider the case $p = 2$ in the present work.

Theorem 5.1. *Let $\mathcal{U}$ be a minimal bounded homogeneous domain and μ a finite positive Borel measure on $\mathcal{U}$. Then the following conditions are all equivalent.*
(a) T_μ is a compact operator on $L_a^2(\mathcal{U})$.
(b) $\widetilde{\mu}(z) \to 0$ as $z \to \partial\mathcal{U}$.
(c) μ is a vanishing Carleson measure for $L_a^2(\mathcal{U})$.
(d) $\widehat{\mu}(z) \to 0$ as $z \to \partial\mathcal{U}$.

Proof. Theorem 3.10 shows $(c) \Longleftrightarrow (d)$. We will prove $(a) \Longrightarrow (b) \Longrightarrow (d)$ and $(c) \Longrightarrow (a)$.

First, we prove that $(a) \Longrightarrow (b)$. By Lemma 3.8, we have $k_z \to 0$ weakly in $L_a^2(\mathcal{U})$ as $z \to \partial\mathcal{U}$. Since T_μ is a compact operator, we have $T_\mu k_z \to 0$ in $L_a^2(\mathcal{U})$. Therefore, we have

$$\widetilde{\mu}(z) = |\langle T_\mu k_z, k_z\rangle| \le \|T_\mu k_z\|_2 \|k_z\|_2 = \|T_\mu k_z\|_2 \longrightarrow 0\ (z \to \partial\mathcal{U}).$$

Next, we prove $(b) \Longrightarrow (d)$. We have already shown that

$$\widehat{\mu}(z) \le M_\rho\, \widetilde{\mu}(z) \tag{5.2}$$

in the proof of Theorem 4.1. Therefore, we have $\widehat{\mu}(z) \to 0$ as $z \to \partial\mathcal{U}$.

Finally, we prove $(c) \Longrightarrow (a)$. First, we prove that $\|T_\mu f\|_{L^2(dV)} \le M_\mu \|f\|_{L^2(d\mu)}$ for any $f \in L_a^2(\mathcal{U})$. Since μ is a Carleson measure, we have $T_\mu f \in L_a^2(\mathcal{U})$ by Theorem 4.1.

Take any $g \in L^2_a(\mathcal{U})$. Then, we have

$$\begin{aligned}\langle T_\mu f, g\rangle &= \int_{\mathcal{U}} \left(\int_{\mathcal{U}} K_{\mathcal{U}}(z,w) f(w)\, d\mu(w) \right) \overline{g(z)}\, dV(z) \\ &= \int_{\mathcal{U}} \left(\int_{\mathcal{U}} K_{\mathcal{U}}(z,w) \overline{g(z)}\, dV(z) \right) f(w)\, d\mu(w) \\ &= \int_{\mathcal{U}} f(w) \overline{g(w)}\, d\mu(w).\end{aligned}$$

Note that we can change the order of integral because (4.5) holds for the case $g \in L^2_a(\mathcal{U})$. Since

$$|\langle T_\mu f, g\rangle| \leq \|f\|_{L^2(d\mu)} \|g\|_{L^2(d\mu)} \leq M_\mu \|f\|_{L^2(d\mu)} \|g\|_{L^2(dV)},$$

we have

$$\|T_\mu f\|_2 \leq M_\mu \|f\|_{L^2(d\mu)}. \tag{5.3}$$

Next, we prove the compactness of T_μ. Take any sequence $\{f_n\}$ such that $f_n \to 0$ weakly in $L^2_a(\mathcal{U})$. Since μ is a vanishing Carleson measure for $L^2_a(\mathcal{U})$, we have $f_n \to 0$ in $L^2_a(\mathcal{U}, d\mu)$. Therefore we have $\|T_\mu f_n\|_2 \to 0$ by (5.3). It means that T_μ is a compact operator on $L^2_a(\mathcal{U})$. □

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